



11.8 Isometric operators

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Introduction

- In this topic, we will
 - Define what it is for a linear operator to be isometric
 - Consider some consequences
 - Look at some useful examples





Definition

- A linear operator $A: \mathcal{U} \rightarrow \mathcal{U}$ is said to be *isometric* if

$$\|A\mathbf{u}\|_2 = \|\mathbf{u}\|_2$$

for all $\mathbf{u} \in \mathcal{U}$

- This means that A does not change the length of any vector
- For example, the identity operator I is isometric,
as is any permutation matrix
- The term *isometric* means *same measure*





Definition

Theorem

The null space of an isometric linear operator A is $\{\mathbf{0}\}$, and thus, for finite-dimensional vector spaces, A is one-to-one and onto and thus invertible.

Proof:

If $\mathbf{u} \in \text{null}(A)$ and A is isometric
then $\|A\mathbf{u}\|_2 = \|\mathbf{u}\|_2 = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Thus, $A\mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$

Thus, the null space is $\{\mathbf{0}\}$.

Thus, if A is an isometric matrix, A is invertible. ■





Definition

Theorem

A finite-dimensional linear operator is isometric if and only if $AA^* = A^*A = I$.

Proof:

If $\|A\mathbf{u}\|_2 = \|\mathbf{u}\|_2$, $\langle A\mathbf{u}, A\mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle$

which is true if and only if $\langle A^*A\mathbf{u}, \mathbf{u} \rangle = \langle I\mathbf{u}, \mathbf{u} \rangle$

If this is true, if and only if the two operators are equal.

Thus, $A^*A = I$.

For finite dimensional linear operators,

this means $AA^* = I$ ■





Definition

Theorem

A finite-dimensional isometric linear operator is normal.

Proof:

From the last theorem, $A^*A = I = A A^*$, so A is normal. ■





Definition

- Up to this point, we have emphasized that the most appropriate interpretation of matrices is where the columns are vectors
- Thus, we often wrote:

$$A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \cdots \ \mathbf{a}_n)$$

- Now we will use $\mathbf{a}_{:,k}$ to represent the k^{th} column, where the vertical ellipsis says “all entries in this column”

$$\mathbf{a}_{:,k} = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \\ \vdots \\ a_{m,k} \end{pmatrix}$$





Definition

- Similarly, we will use $\mathbf{a}_{k,\dots}$ to represent the k^{th} row, where the horizontal ellipsis says “all entries in row”
- Thus, $\mathbf{a}_{k,\dots} = (a_{k,1} \ a_{k,2} \ a_{k,3} \ \cdots \ a_{k,n})$
- Thus, we have $A = (\mathbf{a}_{:,1} \ \mathbf{a}_{:,2} \ \mathbf{a}_{:,3} \ \cdots \ \mathbf{a}_{:,n}) = \begin{pmatrix} \mathbf{a}_{1,\dots} \\ \mathbf{a}_{2,\dots} \\ \vdots \\ \mathbf{a}_{m,\dots} \end{pmatrix}$





Definition

- Also, we will use the adjoint to define:

$$\mathbf{a}_{:,k}^T = \begin{pmatrix} a_{1,k} & a_{2,k} & a_{3,k} & \cdots & a_{m,k} \end{pmatrix}$$

$$\mathbf{a}_{:,k}^* = \begin{pmatrix} a_{1,k}^* & a_{2,k}^* & a_{3,k}^* & \cdots & a_{m,k}^* \end{pmatrix}$$

$$\mathbf{a}_{k,\dots}^T = \begin{pmatrix} a_{k,1} \\ a_{k,2} \\ a_{k,3} \\ \vdots \\ a_{k,n} \end{pmatrix} \quad \mathbf{a}_{k,\dots}^* = \begin{pmatrix} a_{k,1}^* \\ a_{k,2}^* \\ a_{k,3}^* \\ \vdots \\ a_{k,n}^* \end{pmatrix}$$





Definition

- Thus, we have

$$A = \begin{pmatrix} \mathbf{a}_{:,1} & \mathbf{a}_{:,2} & \mathbf{a}_{:,3} & \cdots & \mathbf{a}_{:,n} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{1,\dots} \\ \mathbf{a}_{2,\dots} \\ \vdots \\ \mathbf{a}_{m,\dots} \end{pmatrix}$$
$$A^T = \begin{pmatrix} \mathbf{a}_{:,1}^T \\ \mathbf{a}_{:,2}^T \\ \vdots \\ \mathbf{a}_{:,n}^T \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{1,\dots}^T & \mathbf{a}_{2,\dots}^T & \mathbf{a}_{3,\dots}^T & \cdots & \mathbf{a}_{m,\dots}^T \end{pmatrix}$$
$$A^* = \begin{pmatrix} \mathbf{a}_{:,1}^* \\ \mathbf{a}_{:,2}^* \\ \vdots \\ \mathbf{a}_{:,n}^* \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{1,\dots}^* & \mathbf{a}_{2,\dots}^* & \mathbf{a}_{3,\dots}^* & \cdots & \mathbf{a}_{m,\dots}^* \end{pmatrix}$$





Columns form an orthonormal basis

Theorem

For finite-dimensional vector spaces, if A is isometric, then the columns form an orthonormal basis.

Proof:

Suppose that A is isometric and $A = (\mathbf{a}_{:,1} \ \mathbf{a}_{:,2} \ \mathbf{a}_{:,3} \ \cdots \ \mathbf{a}_{:,n})$

$$A^* A = \begin{pmatrix} \mathbf{a}_{:,1}^* \\ \mathbf{a}_{:,2}^* \\ \vdots \\ \mathbf{a}_{:,n}^* \end{pmatrix} (\mathbf{a}_{:,1} \ \mathbf{a}_{:,2} \ \cdots \ \mathbf{a}_{:,n}) = \begin{pmatrix} \mathbf{a}_{:,1}^* \mathbf{a}_{:,1} & \mathbf{a}_{:,1}^* \mathbf{a}_{:,2} & \cdots & \mathbf{a}_{:,1}^* \mathbf{a}_{:,n} \\ \mathbf{a}_{:,2}^* \mathbf{a}_{:,1} & \mathbf{a}_{:,2}^* \mathbf{a}_{:,2} & \cdots & \mathbf{a}_{:,2}^* \mathbf{a}_{:,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{:,n}^* \mathbf{a}_{:,1} & \mathbf{a}_{:,n}^* \mathbf{a}_{:,2} & \cdots & \mathbf{a}_{:,n}^* \mathbf{a}_{:,n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Thus, the columns form an orthonormal basis. ■





Rows form an orthonormal basis

Theorem

For finite-dimensional vector spaces, if A is isometric, then the rows form an orthonormal basis.

Proof:

Suppose that A is isometric and $A = \begin{pmatrix} \mathbf{a}_{1,\dots} \\ \mathbf{a}_{2,\dots} \\ \vdots \\ \mathbf{a}_{n,\dots} \end{pmatrix}$

$$\begin{pmatrix} \mathbf{a}_{1,\dots} \\ \mathbf{a}_{2,\dots} \\ \vdots \\ \mathbf{a}_{n,\dots} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{1,\dots}^* & \mathbf{a}_{2,\dots}^* & \cdots & \mathbf{a}_{n,\dots}^* \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{1,\dots} \mathbf{a}_{1,\dots}^* & \mathbf{a}_{1,\dots} \mathbf{a}_{2,\dots}^* & \cdots & \mathbf{a}_{1,\dots} \mathbf{a}_{n,\dots}^* \\ \mathbf{a}_{2,\dots} \mathbf{a}_{1,\dots}^* & \mathbf{a}_{2,\dots} \mathbf{a}_{2,\dots}^* & \cdots & \mathbf{a}_{2,\dots} \mathbf{a}_{n,\dots}^* \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{n,\dots} \mathbf{a}_{1,\dots}^* & \mathbf{a}_{n,\dots} \mathbf{a}_{2,\dots}^* & \cdots & \mathbf{a}_{n,\dots} \mathbf{a}_{n,\dots}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Thus, the rows form an orthonormal basis, too. ■





Rows form an orthonormal basis

Theorem

The determinant of a real isometric matrix is ± 1 .

Proof:

$$\det(AA^*) = \det(I) = 1$$

But $\det(AA^*) = \det(A) \det(A^*)$, and $\det(A^*) = \det(A)$

Thus, $\det(A)^2 = 1$, so $\det(A) = \pm 1$. ■





Definition

- If a real square matrix is isometric,
we call it an *orthogonal* matrix
- If a complex square matrix is isometric,
we call it a *unitary* matrix
- In both cases, the rows and columns form an orthonormal basis
for $\mathbf{R}^n$ and $\mathbf{C}^n$, respectively





Definition

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for $\mathbf{R}^n$ and $\mathbf{C}^n$, respectively





Examples

- A few examples examples:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$$





Not everything holds in infinite dimensions...

- Consider the delay operator:

$$Dx = (0, x[0], x[1], x[2], \dots)$$

- This operator is isometric, as it continues to hold all the same entries, but while $ED = I$,

we note that $DEx = (0, x[1], x[2], x[3], \dots)$

- Consequently, $DE \neq I$





Summary

- Following this topic, you now
 - Understand what it means for a linear operator to be isometric
 - Understand that these are invertible in finite dimensional vector spaces
 - Know that the row and column vectors of isometric matrices form orthonormal bases for the corresponding vector spaces
 - Are aware that
 - for real matrices, such operators are called *orthogonal matrices*
 - and for complex matrices, they are called *unitary matrices*





References

- [1] <https://en.wikipedia.org/wiki/Isometry>
- [2] https://en.wikipedia.org/wiki/Orthogonal_matrix
- [3] https://en.wikipedia.org/wiki/Unitary_matrix





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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